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Existence of global attractors for the Benjamin–Bona–Mahony equation in unbounded domains

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Abstract
In this paper, we investigate the asymptotic behavior of the Benjamin–Bona–Mahony equation in unbounded domains. We prove the existence of a global attractor when the equation is defined in a three-dimensional channel. The asymptotic compactness of the solution operator is obtained by the uniform estimates on the tails of solutions.

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1. Introduction

In this paper, we investigate the asymptotic behavior of solutions of the Benjamin–Bona–Mahony (BBM) equation defined in an unbounded domain. Let $Q = \Omega \times \mathbf{R}$ where Ω is a bounded open subset of $\mathbf{R}^2$. Consider the Benjamin–Bona–Mahony equation defined on Q:

$$u_t - \Delta u_t - \nu \Delta u + \nabla \cdot \vec{F}(u) = g(x), \qquad x \in Q, \quad t > 0,$$

where ν is a positive constant, $g \in L^2(Q)$ is given and $\vec{F}$ is a nonlinear vector function satisfying some growth conditions.

The BBM equation was proposed in [12] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions were studied extensively in the literature (see, e.g., [5, 6, 12–14, 16, 19, 24, 25] and the references therein). If the domain is bounded, the existence and regularity of global attractors of the equation were investigated in [4, 15, 17, 39, 40, 43]. If the domain is the entire space $\mathbf{R}^3$, the existence of a *local* attractor for the equation was proved by the authors of [34] under the condition that the external term g *is sufficiently small*. In this paper, we will establish the existence of a *global* attractor for the BBM equation for *any* $g \in L^2(Q)$ when the domain Q is the three-dimensional channel.

Note that the unboundedness of the domain Q introduces a major difficulty for proving the existence of a global attractor because Sobolev embeddings are no longer compact in this case, and hence the asymptotic compactness of the solution operator cannot be obtained by a standard method. An effective way to overcome this difficulty is the energy equation method which was first introduced by Ball in [8, 9] and then employed by several authors to prove the asymptotic compactness of nonlinear dissipative equations in unbounded domains (see, e.g., [18, 20, 22, 27, 28, 32, 44] and the references therein). In this paper, we will use the techniques of uniform estimates on the tails of solutions to establish the asymptotic compactness of the BBM equation in the unbounded channel. This idea was developed in [38] for proving the asymptotic compactness of the reaction–diffusion equation in unbounded domains, and later used by several authors for partial differential equations in [1, 2, 3, 26, 29, 31, 35], and for lattice ordinary differential equations in [10, 11, 23, 37, 41, 42, 45].

This paper is organized as follows. In the next section, we derive uniform estimates on solutions of the BBM equation in the three-dimensional channel when $t \to \infty$, which are necessary for proving the existence of a bounded absorbing set and the asymptotic compactness of the equation. In section 3, we first establish the asymptotic compactness of the solution operator by uniform estimates on the tails of solutions, and then prove the existence of a global attractor.

In what follows, we adopt the following notations. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(Q)$, respectively. The norm of any Banach space X is written as $\|\cdot\|_X$. We also use $\|\cdot\|_p$ to denote the norm of $L^p(Q)$. The letter c is a generic positive constant which may change its value from line to line.

2. Uniform estimates of solutions

In this section, we derive uniform estimates on solutions of the BBM equation in three-dimensional channels for large time. We also establish the uniform estimates on the tails of the solutions for large space variables.

Consider the following problem defined on $Q = \Omega \times \mathbf{R}$, where Ω is a bounded open subset of $\mathbf{R}^2$,

$$u_t - \Delta u_t - \nu\Delta u + \nabla\cdot\vec{F}(u) = g(x), \qquad x \in Q, \quad t > 0, \tag{2.1}$$

with the boundary condition

$$u|_{\partial Q} = 0, \qquad t > 0, \tag{2.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x \in Q, \tag{2.3}$$

where ν is a positive constant, $g \in L^2(Q)$ is given, and $\vec{F}$ is a nonlinear vector function: $\vec{F}(s) = (F_1(s), F_2(s), F_3(s))$ for $s \in \mathbf{R}$. Throughout this paper, we assume that F_k ($k = 1, 2, 3$) is a smooth function satisfying

$$F_k(0) = 0, \qquad |F_k'(s)| \leqslant c_1 + c_2|s|^m, \qquad s \in \mathbf{R}, \tag{2.4}$$

where $0 \leqslant m < 2$. Denote by

$$G_k(s) = \int_0^s F_k(t)\,\mathrm{d}t \qquad \text{and} \qquad f_k(s) = F_k'(s).$$

Then it follows from (2.4) that, for $k = 1, 2, 3$,

$$|f_k(s)| \leqslant c_1 + c_2|s|^m, \quad |F_k(s)| \leqslant c_1|s| + c_2|s|^{m+1} \quad \text{and} \quad |G_k(s)| \leqslant c_1|s|^2 + c_2|s|^{m+2}. \tag{2.5}$$

It is standard to show that, under assumption (2.4), problem (2.1)–(2.3) is well-posed in $H_0^1(Q)$ (see, e.g., [5, 34]), i.e., for every $u_0 \in H_0^1(Q)$, problem (2.1)–(2.3) has a unique solution u such that $u \in C^0\big([0,\infty), H_0^1(Q)\big)$ and $\frac{du}{dt} \in L^\infty\big(0,T; H_0^1(Q)\big)$ for every $T > 0$. Hence, we can associate a semigroup $\{S(t)\}_{t\geqslant 0}$ with problem (2.1)–(2.3) such that for every $t \geqslant 0$, $S(t)$ maps $H_0^1(Q)$ into itself and $S(t)u_0 = u(t)$, the solution of system (2.1)–(2.3). Furthermore, the solution is continuous with respect to the initial condition, which implies that $\{S(t)\}_{t\geqslant 0}$ is a continuous dynamical system on $H_0^1(Q)$. In this paper, we will study the asymptotic behavior of $\{S(t)\}_{t\geqslant 0}$ as $t \to \infty$. We start with the uniform estimates in $H_0^1(Q)$.

Lemma 2.1. *Suppose $g \in L^2(Q)$. Then there exists a positive constant M, depending only on the data (ν, Q, g), such that, for any fixed $R > 0$, the solution u of problem (2.1)–(2.3) with $\|u_0\|_{H_0^1(Q)} \leqslant R$ satisfies*

$$\|u(t)\|_{H_0^1(Q)} \leqslant M, \qquad \forall\, t \geqslant T,$$

where T depends on the data (ν, Q, g) and R.

In this paper, we will frequently use the following Poincare inequality:

$$\|u\| \leqslant \lambda \|\nabla u\|, \qquad \forall\, u \in H_0^1(Q), \tag{2.6}$$

where λ is a positive constant.

Proof. Taking the inner product of (2.1) with u in $L^2(Q)$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|\nabla u\|^2) + \nu\|\nabla u\|^2 + \int_Q (\nabla \cdot \vec{F}(u))u \,\mathrm{d}x = (g, u). \tag{2.7}$$

Let $\vec{G}$ be the antiderivative of $\vec{F}$, i.e.,

$$\vec{G}(s) = \int_0^s \vec{F}(t)\,\mathrm{d}t, \qquad s \in \mathbf{R}. \tag{2.8}$$

Then we have $\nabla \cdot \vec{G}(u) = \vec{F}(u) \cdot \nabla u$ and hence

$$\int_Q (\nabla \cdot \vec{F}(u))u\,\mathrm{d}x = -\int_Q \vec{F}(u)\cdot \nabla u \,\mathrm{d}x = -\int_Q \nabla \cdot \vec{G}(u)\,\mathrm{d}x = 0. \tag{2.9}$$

By (2.6) we find that

$$\nu\|\nabla u\|^2 = \frac{\nu}{2}\|\nabla u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 \geqslant \frac{\nu}{2}\|\nabla u\| + \frac{\nu}{2\lambda^2}\|u\|^2. \tag{2.10}$$

Note that the right-hand side of (2.7) is bounded by

$$|(g, u)| \leqslant \|g\|\|u\| \leqslant \frac{\nu}{4\lambda^2}\|u\|^2 + \frac{\lambda^2}{\nu}\|g\|^2. \tag{2.11}$$

It follows from (2.7)–(2.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|\nabla u\|^2) + \nu\|\nabla u\|^2 + \frac{\nu}{2\lambda^2}\|u\|^2 \leqslant \frac{2\lambda^2}{\nu}\|g\|^2.$$

Let $\alpha = \min\big(\nu, \frac{\nu}{2\lambda^2}\big)$. We find that, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 + \alpha\|u\|_{H^1}^2 \leqslant \frac{2\lambda^2}{\nu}\|g\|^2.$$

Then by Gronwall's lemma we have that

$$\|u(t)\|_{H^1}^2 \leqslant \mathrm{e}^{-\alpha t}\|u_0\|_{H^1}^2 + \frac{2\lambda^2}{\alpha\nu}\|g\|^2 \leqslant \mathrm{e}^{-\alpha t}R^2 + \frac{2\lambda^2}{\alpha\nu}\|g\|^2 \leqslant \frac{3\lambda^2}{\alpha\nu}\|g\|^2, \qquad \forall\, t \geqslant T,$$

where $T = -\frac{1}{\alpha}\ln\big(\frac{\lambda^2\|g\|^2}{R^2\alpha\nu}\big)$. This completes the proof. □

Lemma 2.2. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Then there exists M_1, depending only on the data (ν, Q, g), such that, for any fixed $R > 0$, the solution u of problem (2.1)–(2.3) with $\|u_0\|_{H_0^1(Q)} \leqslant R$ satisfies*

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_{H_0^1(Q)} \leqslant M_1, \qquad \forall\, t \geqslant T_1,$$

where T_1 depends on (ν, Q, g) and R.

Proof. Taking the inner product of (2.1) with u_t in $L^2(Q)$ we get that

$$\|u_t\|^2 + \|\nabla u_t\|^2 + \nu(\nabla u, \nabla u_t) + \int_Q (\nabla \cdot \vec{F}(u)) u_t \,\mathrm{d}x = (g, u_t). \tag{2.12}$$

Note that

$$|(g, u_t)| \leqslant \|g\|\|u_t\| \leqslant \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|g\|^2. \tag{2.13}$$

By lemma 2.1 we find that, for $t \geqslant T$,

$$\nu|(\nabla u, \nabla u_t)| \leqslant \nu\|\nabla u\|\|\nabla u_t\| \leqslant \tfrac{1}{4}\|\nabla u_t\|^2 + \nu^2\|\nabla u\|^2 \leqslant \tfrac{1}{4}\|\nabla u_t\|^2 + \nu^2 M. \tag{2.14}$$

By (2.5) and lemma 2.1, we also have the estimates

$$\begin{aligned}
\left| \int_Q (\nabla \cdot \vec{F}(u)) u_t \,\mathrm{d}x \right| &= \left| \int_Q \vec{F}(u) \cdot \nabla u_t \,\mathrm{d}x \right| \\
&\leqslant c_1 \int_Q |u||\nabla u_t| \,\mathrm{d}x + c_2 \int_Q |u|^{m+1}|\nabla u_t| \,\mathrm{d}x \\
&\leqslant \frac{1}{4}\|\nabla u_t\|^2 + c\|u\|^2 + c\int_Q |u|^{2m+2} \,\mathrm{d}x \\
&\leqslant \frac{1}{4}\|\nabla u_t\|^2 + c\|u\|^2 + c\|u\|_{2m+2}^{2m+2} \\
&\leqslant \frac{1}{4}\|\nabla u_t\|^2 + c\|u\|^2 + c\|u\|_{H^1}^{2m+2} \leqslant \frac{1}{4}\|\nabla u_t\|^2 + c.
\end{aligned} \tag{2.15}$$

Here we used the embedding of $H^1(Q) \hookrightarrow L^p(Q)$ with $2 \leqslant p \leqslant 6$, i.e.,

$$\|u\|_p \leqslant c(p)\|u\|_{H_0^1}, \qquad \forall\, u \in H_0^1(Q), \qquad 2 \leqslant p \leqslant 6,$$

where $c(p)$ is a positive constant depending on p. By (2.12)–(2.15) we obtain

$$\tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|\nabla u_t\|^2 \leqslant c, \qquad \forall\, t \geqslant T,$$

which implies lemma 2.2. The proof is complete. □

Next we derive uniform estimates on the tails of solutions, which are crucial for proving the asymptotic compactness of the solution operator. To this end, for every $x \in Q = \Omega \times \mathbf{R}$, we will write $x = (x_1, x_2, x_3)$ where $(x_1, x_2) \in \Omega$ and $x_3 \in \mathbf{R}$. Given $k > 0$, denote by $Q_k = \{(x_1, x_2, x_3) \in Q\colon |x_3| < k\}$, and $Q \backslash Q_k$ the complement of Q_k.

Note that (2.5) and (2.8) imply that

$$|\vec{G}(s)| \leqslant c_1|s|^2 + c_2|s|^{m+2}, \qquad \text{for} \quad s \in \mathbf{R}, \tag{2.16}$$

which is useful for deriving uniform estimates on the tails of solutions.

Lemma 2.3. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Then given $\epsilon > 0$ and $R > 0$, there exist positive constants T_2 and k_0 such that the solution u of problem (2.1)–(2.3) with $\|u_0\|_{H_0^1(Q)} \leqslant R$ satisfies*

$$\int_{Q\backslash Q_{k_0}} (|u(x,t)|^2 + |\nabla u(x,t)|^2) \,\mathrm{d}x \leqslant \epsilon, \qquad \forall\, t \geqslant T_2,$$

where k_0 depends on (ν, Q, g) and ϵ, T_2 depends on (ν, Q, g), ϵ and R.

Proof. Take a smooth function ϕ such that $0 \leqslant \phi \leqslant 1$ for all $s \in \mathbf{R}$ and

$$\phi(s) = \begin{cases} 0, & \text{if} \quad |s| < 1, \\ 1, & \text{if} \quad |s| > 2. \end{cases} \tag{2.17}$$

Multiplying (2.1) by $\phi^2(\frac{x_3^2}{k^2})u(x,t)$ and then integrating with respect to x on Q, we get

$$\int_Q u_t \phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x - \int_Q \Delta u_t \phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x - \nu \int_Q \Delta u\,\phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x$$
$$+ \int_Q (\nabla \cdot \vec{F}(u))\phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x = \int_Q g\phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x. \tag{2.18}$$

We now estimate each term in (2.18). Note that the first term on the left-hand side of (2.18) is given by

$$\int_Q u_t \phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) u^2\,\mathrm{d}x, \tag{2.19}$$

and the second term is given by

$$\begin{aligned} -\int_Q \Delta u_t\,\phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x &= \int_Q (\nabla u_t \cdot \nabla u)\phi^2\left(\frac{x_3^2}{k^2}\right) \mathrm{d}x + \int_Q u\left(\nabla u_t \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) \mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) |\nabla u|^2\,\mathrm{d}x + \int_Q u\left(\nabla u_t \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) \mathrm{d}x. \end{aligned} \tag{2.20}$$

For the third term on the left-hand side of (2.18) we have

$$-\nu \int_Q \Delta u \phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x = \nu \int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) |\nabla u|^2\,\mathrm{d}x + \nu \int_Q \left(\nabla u \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x. \tag{2.21}$$

The last term on the left-hand side of (2.18) can be written as

$$\int_Q (\nabla \cdot \vec{F}(u))\phi^2\left(\frac{x_3^2}{k^2}\right) u\,\mathrm{d}x = -\int_Q (\vec{F}(u) \cdot \nabla u)\phi^2\left(\frac{x_3^2}{k^2}\right) \mathrm{d}x - \int_Q \left(\vec{F} \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x. \tag{2.22}$$

It follows from (2.18)–(2.22) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) (|u|^2 + |\nabla u|^2)\,\mathrm{d}x\right) + \nu \int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) |\nabla u|^2\,\mathrm{d}x \\ &\quad = -\int_Q \left(\nabla u_t \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x - \nu \int_Q \left(\nabla u \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x \\ &\qquad + \int_Q (\vec{F}(u) \cdot \nabla u)\phi^2\left(\frac{x_3^2}{k^2}\right) \mathrm{d}x + \int_Q \left(\vec{F}(u) \cdot \nabla \phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x \\ &\qquad + \int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) gu\,\mathrm{d}x. \end{aligned} \tag{2.23}$$

We now estimate the right-hand side of (2.23). The first term on the right-hand side of (2.23) is bounded by

$$\left|\int_Q \left(\nabla u_t \cdot \nabla\phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x\right| \leqslant \int_Q |\nabla u_t| \left|2\phi\,\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{2|x_3|}{k^2}|u|\,\mathrm{d}x$$
$$\leqslant \int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |\nabla u_t| \left|2\phi\,\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{2|x_3|}{k^2}|u|\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |\nabla u_t||u|\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\int_Q |\nabla u_t||u|\,\mathrm{d}x \leqslant \frac{c}{k}\|\nabla u_t\|\|u\| \leqslant \frac{c}{k}, \quad \forall\, t \geqslant T, \tag{2.24}$$

where we have used lemmas 2.1 and 2.2. Similarly, for the second term on the right-hand side of (2.23) we have, for $t \geqslant T$,

$$\nu\left|\int_Q \left(\nabla u \cdot \nabla\phi^2\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x\right| \leqslant \nu\int_Q |\nabla u||2\phi| \left|\phi'\left(\frac{x_3^2}{k^2}\right)\right| \left|\frac{2x_3}{k^2}\right| |u|\,\mathrm{d}x$$
$$\leqslant \nu\int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |\nabla u||2\phi| \left|\phi'\left(\frac{x_3^2}{k^2}\right)\right| \left|\frac{2x_3}{k^2}\right| |u|\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |\nabla u||u|\,\mathrm{d}x \leqslant \frac{c}{k}\|\nabla u\|\|u\| \leqslant \frac{c}{k}. \tag{2.25}$$

The third term on the right-hand side of (2.23) is bounded by

$$\left|\int_Q (\vec{F}(u)\cdot\nabla u)\phi^2\left(\frac{x_3^2}{k^2}\right)\mathrm{d}x\right| = \left|\int_Q (\nabla\vec{G}(u))\phi^2\left(\frac{x_3^2}{k^2}\right)\mathrm{d}x\right|$$
$$= \left|\int_Q \vec{G}(u)\cdot\nabla\phi^2\left(\frac{x_3^2}{k^2}\right)\mathrm{d}x\right|$$
$$\leqslant \int_Q \left|\vec{G}(u)\|2\phi\|\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{|2x_3|}{k^2}\,\mathrm{d}x$$
$$\leqslant \int_{k\leqslant|x_3|\leqslant\sqrt{2}k} \left|\vec{G}(u)\|2\phi\|\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{|2x_3|}{k^2}\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\int_Q |\vec{G}(u)|\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\left(\|u\|^2 + \|u\|_{m+2}^{m+2}\right) \leqslant \frac{c}{k}\left(\|u\|^2 + \|u\|_{H^1}^{m+2}\right) \leqslant \frac{c}{k}, \quad \forall\, t \geqslant T, \tag{2.26}$$

where we have used (2.16) and lemma 2.1. We now estimate the fourth term on the right-hand side of (2.23) as follows:

$$\left|\int_Q \left(\vec{F}(u)\cdot\nabla\phi\left(\frac{x_3^2}{k^2}\right)\right) u\,\mathrm{d}x\right| \leqslant \int_Q |\vec{F}(u)| \left|2\phi\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{2|x_3|}{k^2}|u|\,\mathrm{d}x$$
$$\leqslant \int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |\vec{F}(u)| \left|2\phi\phi'\left(\frac{x_3^2}{k^2}\right)\right| \frac{2|x_3|}{k^2}|u|\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\int_Q |\vec{F}(u)||u|\,\mathrm{d}x \leqslant \frac{c}{k}\int_Q (|u|^2 + |u|^{m+2})\,\mathrm{d}x$$
$$\leqslant \frac{c}{k}\left(\|u\|^2 + \|u\|_{m+2}^{m+2}\right) \leqslant \frac{c}{k}\left(\|u\|^2 + \|u\|_{H^1}^{m+2}\right) \leqslant \frac{c}{k}, \tag{2.27}$$

for all $t \geqslant T$. The last term on the right-hand side (2.23) is bounded by, for all $t \geqslant T$,

$$\left|\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right) gu\,\mathrm{d}x\right| = \left|\int_{|x_3|\geqslant k} \phi^2\left(\frac{x_3^2}{k^2}\right) g(x)u(x)\,\mathrm{d}x\right|$$
$$\leqslant \left(\int_{|x_3|\geqslant k} |u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x_3|\geqslant k} \phi^4\left(\frac{x_3^2}{k^2}\right) g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\leqslant \|u\|\left(\int_{|x_3|\geqslant k} g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant c\left(\int_{|x_3|\geqslant k} g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \qquad (2.28)$$

Finally, it follows from (2.23)–(2.28) that, for $t \geqslant T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2+|\nabla u|^2)\,\mathrm{d}x\right) + 2\nu\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x$$
$$\leqslant \frac{c}{k} + c\left(\int_{|x_3|\geqslant k} g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \qquad (2.29)$$

In order to apply Gronwall's lemma to (2.29), we need to deal with the second term on the left-hand side again. Note that, for $t \geqslant T$,

$$\int_Q \left|\nabla\left(\phi\left(\frac{x_3^2}{k^2}\right)u\right)\right|^2\mathrm{d}x = \int_Q \left|u\nabla\phi\left(\frac{x_3^2}{k^2}\right) + \phi\left(\frac{x_3^2}{k^2}\right)\nabla u\right|^2\mathrm{d}x$$
$$\leqslant 2\int_Q |u|^2\left|\nabla\phi\left(\frac{x_3^2}{k^2}\right)\right|^2\mathrm{d}x + 2\int_Q\left|\phi\left(\frac{x_3^2}{k^2}\right)\right|^2|\nabla u|^2\,\mathrm{d}x$$
$$\leqslant 2\int_{k\leqslant|x_3|\leqslant\sqrt{2}k} |u|^2\left|\phi'\left(\frac{x_3^2}{k^2}\right)\right|^2\frac{|2x_3|^2}{k^4}\,\mathrm{d}x + 2\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x$$
$$\leqslant \frac{c}{k^2}\int_Q |u|^2\,\mathrm{d}x + 2\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x$$
$$\leqslant \frac{c}{k^2} + 2\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x. \qquad (2.30)$$

Since $u \in H_0^1(Q)$ we have $\phi^2(\frac{x_3^2}{k^2})u \in H_0^1(Q)$ and hence by (2.6) and (2.30) we get

$$\left\|\phi\left(\frac{x_3^2}{k^2}\right)u\right\|^2 \leqslant \lambda^2\left\|\nabla\left(\phi\left(\frac{x_3^2}{k^2}\right)u\right)\right\|^2 \leqslant \frac{\lambda^2 c}{k^2} + 2\lambda^2\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x,$$

i.e.,

$$\frac{1}{2\lambda^2}\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)u^2\,\mathrm{d}x - \frac{c}{2k^2} \leqslant \int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x.$$

Then we find that

$$2\nu\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x \geqslant \nu\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x + \frac{\nu}{2\lambda^2}\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)u^2\,\mathrm{d}x - \frac{c\nu}{2k^2}. \qquad (2.31)$$

By (2.29) and (2.31) we obtain, for $t \geqslant T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2+|\nabla u|^2)\,\mathrm{d}x\right) + \nu\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)|\nabla u|^2\,\mathrm{d}x + \frac{\nu}{2\lambda^2}\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)u^2\,\mathrm{d}x$$
$$\leqslant \frac{c\nu}{2k^2} + \frac{c}{k} + c\left(\int_{|x_3|\geqslant k} g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $\alpha = \min\left(\nu, \frac{\nu}{2\lambda^2}\right)$. We get, for $t \geqslant T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2 + |\nabla u|^2)\,\mathrm{d}x\right) + \alpha\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2 + |\nabla u|^2)\,\mathrm{d}x$$
$$\leqslant \frac{c}{k} + \frac{c}{k^2} + c\left(\int_{|x_3|\geqslant k} g^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.32}$$

Since $g \in L^2(Q)$, the right hand side of (2.23) goes to zero as $k \to \infty$, and hence there is $k_1 > 0$, such that for every $k \geqslant k_1$, and $t \geqslant T$, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2 + |\nabla u|^2)\,\mathrm{d}x\right) + \alpha\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u|^2 + |\nabla u|^2)\,\mathrm{d}x < \varepsilon.$$

By Gronwall's lemma we obtain, for $k \geqslant k_1$, and $t \geqslant T$,

$$\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u(t)|^2 + |\nabla u(t)|^2)\,\mathrm{d}x \leqslant \mathrm{e}^{-\alpha(t-T)}\left(\int_Q \phi^2\left(\frac{x_3^2}{k^2}\right)(|u(T)|^2 + |\nabla u(T)|^2\,\mathrm{d}x\right) + \frac{\varepsilon}{\alpha}$$
$$\leqslant \mathrm{e}^{-\alpha(t-T)}\|u(T)\|_{H^1}^2 + \frac{\varepsilon}{\alpha} \leqslant \mathrm{e}^{-\alpha(t-T)}M^2 + \frac{\varepsilon}{\alpha} \leqslant \frac{2\varepsilon}{\alpha}, \tag{2.33}$$

for all $t \geqslant T^*$, where $T^* = T - \frac{1}{\alpha}\ln\frac{\varepsilon}{\alpha M^2}$. Note that (2.33) implies, for $k \geqslant k_1$ and $t \geqslant T^*$,

$$\int_{|x_3|\geqslant\sqrt{2}k}(|u(t)|^2 + |\nabla u(t)|^2)\,\mathrm{d}x \leqslant \frac{2\varepsilon}{\alpha},$$

and then lemma 2.3 follows immediately. This completes the proof. □

In order to get the asymptotic compactness of the solution operator, we also need to establish the uniform estimates of the solutions on bounded domains. To this end, we define $\psi = 1 - \phi$ where ϕ is the function given in (2.17). Fix $k \geqslant 1$ and let $v(x,t) = \psi\left(\frac{x_3^2}{k^2}\right)u(x,t)$. Then by lemma 2.1 we have $v \in H_0^1(Q_{2k})$ and

$$\|v(t)\|_{H_0^1(Q_{2k})} \leqslant c, \qquad \forall\, t \geqslant T, \tag{2.34}$$

where c is a positive constant independent of k. Note that

$$v_t = \psi u_t, \tag{2.35}$$
$$\Delta v = (\Delta\psi)u + 2\nabla\psi\cdot\nabla u + \psi\Delta u, \tag{2.36}$$
$$\Delta v_t = (\Delta\psi)u_t + 2\nabla\psi\cdot\nabla u_t + \psi\Delta u_t. \tag{2.37}$$

By (2.36) and (2.37) we find that

$$\psi\Delta u = \Delta v - u\Delta\psi - 2\nabla\psi\cdot\nabla u \tag{2.38}$$

and

$$\psi\Delta u_t = \Delta v_t - u_t\Delta\psi - 2\nabla\psi\cdot\nabla u_t. \tag{2.39}$$

Multiplying (2.1) by ψ we obtain

$$\psi u_t - \psi\Delta u_t - \nu\psi\Delta u + \psi\nabla\cdot\vec{F}(u) = \psi g. \tag{2.40}$$

Substituting (2.35), (2.38) and (2.39) into (2.40) we get

$$v_t - \Delta v_t - \nu\Delta v = \psi g - \psi\nabla\cdot\vec{F}(u) - u_t\Delta\psi - \nu u\Delta\psi - 2\nabla\psi\cdot\nabla u_t - 2\nu\nabla\psi\cdot\nabla u. \tag{2.41}$$

Consider the eigenvalue problem:

$$-\Delta u = \lambda u \qquad \text{in} \quad Q_{2k}, \qquad u|_{\partial Q_{2k}} = 0. \tag{2.42}$$

Then problem (2.42) has a family of eigenfunctions $\{\omega_j\}_{j=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ such that $\{\omega_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(Q_{2k})$ and

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to \infty \qquad \text{as} \quad j \to \infty.$$

Given n, let $X_n = \operatorname{span}\{\omega_1, \dots, \omega_n\}$ and $P_n : L^2(Q_{2k}) \to X_n$ be the projection operator. We now establish the following estimates.

Lemma 2.4. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Then for every $\varepsilon > 0$ and $k \geqslant 1$, there exists N, depending on k and ε, such that for all $n \geqslant N$,*

$$\|(I - P_n)v(t)\|_{H_0^1(Q_{2k})} \leqslant \varepsilon, \qquad \forall\, t \geqslant T,$$

where T depends on (ν, Q, g), k, ε and R when $\|u_0\|_{H_0^1} \leqslant R$.

Proof. Let $v = v_{n,1} + v_{n,2}$ where $v_{n,1} = P_n v$ and $v_{n,2} = v - v_{n,1}$. Applying $I - P_n$ to (2.41) and then taking the inner product of resulting equation with $v_{n,2}$ in $L^2(Q_{2k})$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v_{n,2}\|^2 + \|\nabla v_{n,2}\|^2) + \nu\|\nabla v_{n,2}\|^2 = (H(u), v_{n,2}), \tag{2.43}$$

where $H(u)$ is the right-hand side of (2.41) , i.e.

$$H(u) = \psi g - \psi\nabla \cdot \vec{F}(u) - u_t\Delta\psi - \nu u\Delta\psi - 2\nabla\psi \cdot \nabla u_t - 2\nu\nabla\psi \cdot \nabla u. \tag{2.44}$$

We now estimate the right-hand side of (2.43). Note that the nonlinear term in $(H(u), v_{n,2})$ is bounded by

$$\begin{aligned}
|(\psi\nabla \cdot \vec{F}(u), v_{n,2})| &= \left| \int_{Q_{2k}} \psi\left(\frac{x_3^2}{k^2}\right)\left(\sum_{k=1}^{3} F_k^{'}(u) \cdot \frac{\partial u}{\partial x_k}\right) v_{n,2}\,\mathrm{d}x \right| \\
&\leqslant c\int_{Q_{2k}} \sum_{k=1}^{3} |f_k(u)||\nabla u||v_{n,2}|\,\mathrm{d}x \\
&\leqslant c\int_{Q_{2k}} |\nabla u||v_{n,2}|\,\mathrm{d}x + c\int_{Q_{2k}} |u|^m|\nabla u||v_{n,2}|\,\mathrm{d}x \\
&\leqslant c\|\nabla u\|\|v_{n,2}\| + c\|\nabla u\|\|u\|_6^m\|v_{n,2}\|_s,
\end{aligned} \tag{2.45}$$

where $s = \frac{6}{3-m}$. The second inequality above is obtained by (2.5), and the last one by Holder inequality due to $\frac{1}{2} + \frac{1}{s} + \frac{m}{6} = 1$. On the other hand, by the Nirenberg–Gagliardo inequality

$$\|w\|_s \leqslant c\|w\|_{H^1}^{\theta}\|w\|^{1-\theta},$$

where $\theta = \frac{m}{2} \in [0, 1)$, we find that

$$\|v_{n,2}\|_s \leqslant c\|v_{n,2}\|_{H^1}^{\theta}\|v_{n,2}\|^{1-\theta} \leqslant c\lambda_{n+1}^{\frac{1}{2}(\theta-1)}\|v_{n,2}\|_{H^1} \leqslant c\lambda_{n+1}^{\frac{1}{2}(\theta-1)}\|\nabla v_{n,2}\|. \tag{2.46}$$

It follows from (2.45) and (2.46) and lemma 2.1 that, for all $t \geqslant T$,

$$\begin{aligned}
|(\psi\nabla \cdot \vec{F}(u), v_{n,2})| &\leqslant c\|\nabla u\|\|v_{n,2}\| + c\|\nabla u\|\|u\|_{H^1}^m\lambda_{n+1}^{\frac{1}{2}(\theta-1)}\|\nabla v_{n,2}\| \\
&\leqslant c\|v_{n,2}\| + c\lambda_{n+1}^{\frac{1}{2}(\theta-1)}\|\nabla v_{n,2}\| \leqslant c\lambda_{n+1}^{-\frac{1}{2}}\|\nabla v_{n,2}\| + c\lambda_{n+1}^{\frac{1}{2}(\theta-1)}\|\nabla v_{n,2}\| \\
&\leqslant \frac{\nu}{8}\|\nabla v_{n,2}\|^2 + c\left(\lambda_{n+1}^{-1} + \lambda_{n+1}^{-(1-\theta)}\right).
\end{aligned} \tag{2.47}$$

We now use lemma 2.2 to estimate the term $(u_t \Delta\psi, v_{n,2})$. Note that, for $t \geqslant T$,

$$
\begin{aligned}
|(u_t \Delta\psi, v_{n,2})| &= \left| \int_{Q_{2k}} u_t v_{n,2} \left(\frac{2}{k^2} \psi' \left(\frac{x_3^2}{k^2} \right) + \frac{4x_3^2}{k^4} \psi'' \left(\frac{x_3^2}{k^2} \right) \right) \mathrm{d}x \right| \\
&\leqslant \frac{c}{k^2} \int_{Q_{2k}} |u_t||v_{n,2}| \, \mathrm{d}x \leqslant \frac{c}{k^2} \|u_t\| \|v_{n,2}\| \leqslant \frac{c}{k^2} \|v_{n,2}\| \\
&\leqslant \frac{c}{k^2} \lambda_{n+1}^{-\frac{1}{2}} \|\nabla v_{n,2}\| \leqslant \frac{\nu}{8} \|\nabla v_{n,2}\|^2 + c\lambda_{n+1}^{-1}.
\end{aligned} \tag{2.48}
$$

Similar to (2.48), all other terms in $(H(u), v_{n,2})$ are also bounded by $\frac{1}{8}\nu\|\nabla v_{n,2}\|^2 + c\lambda_{n+1}^{-1}$, which along with (2.43), (2.47) and (2.48) implies that, for all $t \geqslant T$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|v_{n,2}\|^2 + \|\nabla v_{n,2}\|^2) + \nu\|\nabla v_{n,2}\|^2 \leqslant c\lambda_{n+1}^{-1} + c\lambda_{n+1}^{-(1-\theta)}. \tag{2.49}
$$

Since $1 - \theta > 0$ and $\lambda_n \to \infty$, given $\varepsilon > 0$, there is $N = N(\varepsilon)$ such that for an $n \geqslant N$:

$$
\lambda_n \geqslant 1 \qquad \text{and} \qquad c\lambda_{n+1}^{-1} + c\lambda_{n+1}^{-(1-\theta)} < \varepsilon. \tag{2.50}
$$

By (2.49) and (2.50) we get that, for all $n \geqslant N$ and $t \geqslant T$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|v_{n,2}\|^2 + \|\nabla v_{n,2}\|^2) + \nu\|\nabla v_{n,2}\|^2 \leqslant \varepsilon. \tag{2.51}
$$

Note that

$$
\nu\|\nabla v_{n,2}\|^2 \geqslant \frac{\nu}{2}\|\nabla v_{n,2}\|^2 + \frac{\nu}{2}\lambda_{n+1}\|v_{n,2}\|^2 \geqslant \frac{\nu}{2}(\|v_{n,2}\|^2 + \|\nabla v_{n,2}\|^2),
$$

which along with (2.51) shows that for $n \geqslant N$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v_{n,2}\|_{H^1}^2 + \frac{1}{2}\nu\|v_{n,2}\|_{H^1}^2 \leqslant \varepsilon, \qquad \forall\, t \geqslant T.
$$

By Gronwall's lemma, we find that, for all $n \geqslant N$ and $t \geqslant T$,

$$
\begin{aligned}
\|v_{n,2}(t)\|_{H^1(Q_{2k})}^2 &\leqslant \mathrm{e}^{-\frac{1}{2}\nu(t-T)}\|v_{n,2}(T)\|_{H^1}^2 + \frac{2\varepsilon}{\nu}, \\
&\leqslant \mathrm{e}^{-\frac{1}{2}\nu(t-T)}\|v(T)\|_{H^1}^2 + \frac{2\varepsilon}{\nu} \leqslant c e^{-\frac{1}{2}\nu(t-T)} + \frac{2\varepsilon}{\nu} \leqslant \frac{3\varepsilon}{\nu},
\end{aligned}
$$

for all $t \geqslant T - \frac{2}{\nu} \ln \frac{\varepsilon}{\nu c}$. This completes the proof. □

Lemma 2.5. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Let $\{u_{0,l}\}_{l=1}^{\infty}$ be bounded in $H_0^1(Q)$ and $t_l \to \infty$ as $l \to \infty$. If $u_l(t)$ is the solution of problem (2.1)–(2.3) with the initial condition $u_{0,l}$, and*

$$
v_l(x,t) = \psi\left(\frac{x_3^2}{k^2}\right) u_l(x,t), \qquad l = 1, 2, \ldots, \tag{2.52}
$$

where $k \geqslant 1$ is fixed. Then the sequence $\{v_l(t_l)\}_{l=1}^{\infty}$ has a convergent subsequence in $H_0^1(Q)$.

Proof. Since $\{u_{0,l}\}$ is bounded in $H_0^1(Q)$, there is a positive constant c such that

$$
\|u_{0,l}\|_{H_0^1(Q)} \leqslant c, \qquad l = 1, 2, \ldots. \tag{2.53}
$$

By (2.53) and lemma 2.1 we find that there is $T_1 > 0$ such that for all $t \geqslant T_1$:

$$
\|u_l(t)\|_{H_0^1(Q)} \leqslant c, \qquad l = 1, 2, \ldots. \tag{2.54}
$$

Since $t_l \to \infty$, there is $L > 0$ such that $t_l \geqslant T_1$ for all $l \geqslant L$, and hence by (2.54) we have

$$
\|u_l(t_l)\|_{H_0^1(Q)} \leqslant c, \qquad \forall\, l \geqslant L. \tag{2.55}
$$

By (2.52) and (2.55) we get

$$\|v_l(t_l)\|_{H_0^1(Q)} \leqslant c, \qquad \forall\, l \geqslant L. \tag{2.56}$$

Note that $v_l(x,t) = 0$ for $x \notin Q_{2k}$. Therefore (2.56) implies that

$$\|v_l(t_l)\|_{H_0^1(Q_{2k})} \leqslant c, \qquad \forall\, l \geqslant L. \tag{2.57}$$

Given $\varepsilon > 0$, by lemma 2.4 there are $N = N(k,\varepsilon)$ and $T_2 = T_2(k,\varepsilon)$ such that

$$\|(I - P_N)v_l(t)\|_{H_0^1(Q_{2k})} \leqslant \frac{\varepsilon}{3}, \qquad \forall\, t \geqslant T_2. \tag{2.58}$$

Since $t_l \to \infty$, there is $L_1 > 0$ such that $t_l \geqslant T_2$ for all $l \geqslant L_1$. In this case we obtain from (2.58) that

$$\|(I - P_N)v_l(t_l)\|_{H_0^1(Q_{2k})} \leqslant \frac{\varepsilon}{3}, \qquad \forall\, t \geqslant T_2. \tag{2.59}$$

Let $L_3 = \max\{L, L_1\}$. Then by (2.57) we find that $\{P_N v_l(t_l)\}_{l=L_3}^{\infty}$ is bounded in the finite-dimensional space $P_N H_0^1(Q_{2k})$ and hence is precompact. In other words, for the given ε , there is a finite subset $\left\{v_{l1}(t_{l_1}) \cdots v_{l_r}\left(t_{l_r}\right)\right\}$ of $\{v_l(t_l)\}_{l=L_3}^{\infty}$ such that $\left\{P_N v_{l1}\left(t_{l_1}\right) \cdots P_N v_{l_r}\left(t_{l_r}\right)\right\}$ is an $\frac{\varepsilon}{3}$-net of $\{P_N v_l(t_l)\}_{l=L_3}^{\infty}$ in $P_N H_0^1(Q_{2k})$, which along with (2.59) shows that $\left\{v_{l1}\left(t_{l_1}\right) \cdots v_{l_r}\left(t_{l_r}\right)\right\}$ is an ε-net of $\{v_l(t_l)\}_{l=L_3}^{\infty}$ in $H_0^1(Q_{2k})$. Note that $v_l(x,t_l) = 0$ for $x \notin Q_{2k}$ and hence $\left\{v_{l1}\left(t_{l_1}\right) \cdots v_{l_r}\left(t_{l_r}\right)\right\}$ is also an ε-net of $\{v_l(t_l)\}_{l=L_3}^{\infty}$ in $H_0^1(Q)$, that is, for every $\varepsilon > 0$, the sequence $\{v_l(t_l)\}_{l=1}^{\infty}$ has a finite ε-net in $H_0^1(Q)$. Therefore, $\{v_l(t_l)\}_{l=1}^{\infty}$ is precompact in $H_0^1(Q)$. This completes the proof. □

3. Existence of global attractors

In this section, we prove the existence of global attractors for problem (2.1)–(2.3) in $H_0^1(Q)$. To this end, we need to establish the asymptotic compactness of the solution operator which is stated as follows.

Lemma 3.1. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Let $\{u_{0,l}\}_{l=1}^{\infty}$ be bounded in $H_0^1(Q)$ and $t_l \to \infty$. Then $\{S(t_l)u_{0,l}\}$ has a convergent subsequence in $H_0^1(Q)$.*

Proof. Let $u_l(t) = S(t)u_{0,l}$. Since $\{u_{0,l}\}$ is bounded in $H_0^1(Q)$, there is $R > 0$ such that

$$\|u_{0,l}\|_{H_0^1(Q)} \leqslant R, \qquad l = 1, 2, \ldots. \tag{3.1}$$

Then by lemma 2.3, given $\varepsilon > 0$, there are k_0 and T_1 such that

$$\int_{Q \setminus Q_{k_0}} (|u_l(t)|^2 + |\nabla u_l(t)|^2)\,\mathrm{d}x \leqslant \varepsilon, \qquad \forall\, t \geqslant T_1. \tag{3.2}$$

Since $t_l \to \infty$, there is $L_1 > 0$ such that $t_l \geqslant T_1$ for all $l \geqslant L_1$, and hence by (2.51) we get

$$\int_{Q \setminus Q_{k_0}} (|u_l(t_l)|^2 + |\nabla u_l(t_l)|^2)\,\mathrm{d}x \leqslant \varepsilon, \qquad \forall\, l \geqslant L_1. \tag{3.3}$$

Denote by

$$v_l(x,t) = \psi\left(\frac{x_3^2}{k_0^2}\right) u_l(x,t).$$

Then by lemma 2.5, $\{v_l(t_l)\}$ has a convergent subsequence $\left\{v_{l_n}(t_{l_n})\right\}$ in $H_0^1(Q)$. Note that $\left\{v_{l_n}(t_{l_n})\right\}$ is a Cauchy sequence in $H_0^1(Q)$ and hence also a Cauchy sequence in $H_0^1\left(Q_{k_0}\right)$. By definition, $v_{l_n}\left(t_{l_n}\right) = u_{l_n}\left(t_{l_n}\right)$ on Q_{k_0} and therefore $\left\{u_{l_n}\left(t_{l_n}\right)\right\}$ is a Cauchy sequence in $H^1\left(Q_{k_0}\right)$, which along with (3.3) shows that $\{u_{l_n}(t_{l_n})\}$ is a Cauchy sequence in $H_0^1(Q)$. This completes the proof. □

We are now ready to prove the existence of a global attractor for the Benjamin–Bona–Mahony equation.

Theorem 3.2. *Suppose $g \in L^2(Q)$ and (2.4) is satisfied. Then problem (2.1)–(2.3) has a global attractor $\mathcal{A}$ in $H_0^1(Q)$, which is compact and invariant and attracts every bounded set with respect to the norm of $H_0^1(Q)$.*

Proof. By lemma 2.1, the dynamical system $\{S(t)\}_{t\geqslant 0}$ has a bounded absorbing set in $H_0^1(Q)$, and by lemma 3.1, $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact. Then the existence of a global attractor follows immediately from the standard attractor theory (see, e.g., [7, 8, 21, 30, 33, 36]). The proof is complete. □

4. Discussion

In this paper, we have proved the existence of a global attractor for the Benjamin–Bona–Mahony equation defined in a three-dimensional unbounded channel. The attractor is a compact subset of the phase space which is invariant and attracts all solutions when time goes to infinity. This implies that the permanent states of the physical system after a short transient period are governed by the global attractor. The existence and the complicated structure of the global attractor are responsible for the chaotic and turbulent behavior of the flow described by the Benjamin–Bona–Mahony equation. Studying the dynamical behavior of the flow on the global attractor is a necessary step for better understanding the turbulent mechanism of the physical system. The structure of the global attractor not only determines the chaotic behavior of the flow, but also determines the level of complexity of the physical phenomena. The global attractor obtained in this paper is compact and invariant, and hence, its fractal dimension is likely to be finite. The dimension of the attractor will be addressed in future work. If the attractor is finite-dimensional, then the turbulent motion of the flow will be governed by a finite number of degrees of freedom, and in this case, we may reduce the study of the infinite-dimensional flow to a finite-dimensional system.

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